The local & global Hitchin map

(Notes by Nikolay)

I] § Hamiltonian Reduction

Let K = algebraic group / $\mathbb{C}$, acting on a smooth scheme $Y/\mathbb{C}$.
Consider the quotient (stack) $\mathcal{Y} = K\backslash Y$. Assume $\mathcal{Y}$ good: ($\dim T^*_{\mathcal{Y}} = 2\dim \mathcal{Y}$, so $T^*_{\mathcal{Y}}$ is classical algebraic stack, not a derived stack).

Differentiating the $K \xrightarrow{act} \mathrm{Aut}(Y)$ map induces the "comoment map"

$$\mu^*: \underset{\substack{\parallel \\ \mathrm{Lie}(K)}}{\mathfrak{k}} \longrightarrow \underset{\substack{\parallel \\ \mathrm{Vect}(Y)}}{T_Y}$$

Dually, have the "moment map"

$$\mu : T^*_Y \longrightarrow \mathfrak{k}^*$$

Then, $\boxed{T^*_{\mathcal{Y}} = K\backslash \mu^{-1}(0)}$ and we say $T^*_{\mathcal{Y}}$ is obtained from T^*_Y via Hamiltonian Reduction.

We may summarize this construction via the following fiber square of (underived) stacks:

$$\begin{tikzcd} & T^*_Y \arrow[d, "\mu"] \\ 0 \arrow[r] & \mathfrak{k}^* \end{tikzcd}$$

Both maps K-equivariant

$$\begin{tikzcd} T^*_{\mathcal{Y}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T^*_Y/K \arrow[d] \\ 0/K \arrow[r] & \mathfrak{k}^*/K \end{tikzcd}$$

In fact, $\Gamma(T^*\mathcal{Y}, \mathcal{O}_{T^*\mathcal{Y}})$ carries a Poisson structure induced by that of T^*_Y:

Let $\pi : Y \longrightarrow \mathcal{Y} = K\backslash Y$ be the projection map. Then have exact sequence of $\mathcal{O}_Y$-modules:

$$T_{Y/\mathcal{Y}} \longrightarrow T_Y \longrightarrow \pi^* T_{\mathcal{Y}} \longrightarrow 0$$

relative tangent sheaf, isomorphic to $\underset{\mathrm{Lie}(K)}{\mathfrak{k}} \otimes \mathcal{O}_Y$ (Recall Ginzburg's talk & note on tangent complex).

Remark (to make Justin happy): We may remove goodness assumption on $\mathcal{Y}$ and replace the above with an exact triangle in the derived stacks world. Then e.g. $\pi^* T_{pt/K} = $ "$pt \overset{\mathbb{L}}{\otimes}_{\mathfrak{k}^*} pt / K$", the derived self-intersection of 0 over k^* mod K.

$$\Rightarrow \boxed{\pi^* \mathrm{Sym}\, T_Y \simeq \mathrm{Sym}\, T_Y / (\mathrm{Sym}\, T_Y)\cdot k}$$

Is not Poisson alg, but taking K-invariants is!

← Is precisely the def. of the Y-points of cotangent stack T^*Y.

Let $I^{cl} := \mathrm{Sym}\, T_Y \cdot k$

$$\tilde{P}^{cl} = \{ \xi \in \mathrm{Sym}\, T_Y \mid \{\xi, I^{cl}\} \subseteq I^{cl} \}$$ = Poisson normalizer of I^{cl}

$$\Rightarrow \boxed{P_y := \left(\tilde{P}^{cl}/I^{cl}\right)^{\pi_0(K)} = \left(\mathrm{Sym}\, T_Y / \mathrm{Sym}\, T_Y \cdot k\right)^K}$$

is a sheaf of Poisson algebras

The embedding of sheaf of commutative algebras

$$P_y \hookrightarrow \mathrm{Sym}\, T_y$$

induces an isomorphism of global sections

$$\Gamma(y, P_y) \xrightarrow{\sim} \Gamma(y, \mathrm{Sym}\, T_y) \simeq \Gamma(T^*y, \mathcal{O}_{T^*y})$$

Can quantize! ("Quantum Hamiltonian Reduction")

Consider $I := D_Y \cdot T_{Y/y}$

$$\tilde{D}_Y = \{ u \in D_Y \mid [u, I] \subseteq I \}$$

$$D_y := \tilde{D}_Y / I,$$ is (almost-commutative) algebra,

comes equipped with "principal symbol map"

$$\boxed{\sigma_y : \mathrm{gr}\, D_y \hookrightarrow P_y}$$

embedding of Poisson algebras.

§ Harish-Chandra Pairs

<u>Def</u> A <u>HC pair</u> $(\mathfrak{g}, K)$ is $\mathfrak{g}$ = Lie algebra, K = algebraic gp, $K \curvearrowright \mathfrak{g}$, & $\mathfrak{k} \hookrightarrow \mathfrak{g}$ embedding of Lie algebras which commutes with the K-action.

Suppose Y is smooth scheme w/ $(\mathfrak{g}, K)$-action, i.e
① $K \curvearrowright Y$ & ② $\mathfrak{g} \to \Gamma(Y, T_Y)$ is K-equivariant.

Then we again have a fiber product diagram of stacks:

$$\begin{tikzcd}
T^*_Y \arrow[r] \arrow[d] \arrow[dd, bend right] & T^*_Y/K \arrow[d] & \\
(\mathfrak{g}/\mathfrak{k})^*/K \arrow[r] \arrow[d] \arrow[ur, phantom, "\lrcorner"] & \mathfrak{g}^*/K \arrow[d] & \text{stacky quotient} \\
(\mathfrak{g}/\mathfrak{k})^*/\!/K \arrow[r] & \mathfrak{g}^*/\!/K & \text{GIT quotient}
\end{tikzcd}$$

This will be the <u>local</u> <u>Hitchin map</u>! (After using the identification in Lemma: "$\mathcal{K}_3$ is isomorphism").

Again, now we quantize!

<u>Define</u>

$$P_{(\mathfrak{g},K)} := \mathrm{Sym}(\mathfrak{g}/\mathfrak{k})^K = \Big(\overbrace{\tilde{P}_{(\mathfrak{g},K)}}^{\text{poisson normalizer}} \Big/ \overbrace{I^{cl}_{(\mathfrak{g},K)}}^{(\mathrm{Sym}\,\mathfrak{g})\cdot\mathfrak{k}}\Big)^{\pi_0(K)}$$

(Quantum) Ham. Red. of $\mathrm{Sym}\,\mathfrak{g}$, resp. $U\mathfrak{g}$, along K action.

$$D_{(\mathfrak{g},K)} := \big(U\mathfrak{g}/U\mathfrak{g}\cdot\mathfrak{k}\big)^K = \Big(\underbrace{\tilde{D}_{(\mathfrak{g},K)}}_{[\,,\,]\text{ normalizer}} \Big/ \underbrace{I_{(\mathfrak{g},K)}}_{(U\mathfrak{g})\cdot\mathfrak{k}}\Big)^{\pi_0(K)}$$

Have (local) symbol map

$$\sigma_{(\mathfrak{g},K)} : \mathrm{gr}\, D_{(\mathfrak{g},K)} \longrightarrow P_{(\mathfrak{g},K)}$$

Next, the moment map actions of $(\mathfrak{g},K)$ on Y induce:

$$\mathrm{Sym}\,\mathfrak{g} \to \mathrm{Sym}\,T_Y \quad \rightsquigarrow \quad h^{cl} : P_{(\mathfrak{g},K)} \to \Gamma(Y, P_Y)$$

(mod out $\mathrm{Sym}(\mathfrak{g})\mathfrak{k}$, take K-invariants)

$$U\mathfrak{g} \to \mathcal{D}_Y \quad \rightsquigarrow \quad h : D_{(\mathfrak{g},K)} \to \Gamma(Y, D_Y)$$

These all fit together nicely into the commutative diagram: of Poisson algebras

$$\begin{tikzcd} \mathrm{gr}\, D_{(\mathfrak{g},K)} \arrow[r, "h"] \arrow[d, "\sigma_{(\mathfrak{g},K)}"'] & \mathrm{gr}\, \Gamma(\mathcal{Y}, D_{\mathcal{Y}}) \arrow[d, "\sigma_{\mathcal{Y}}"] \\ P_{(\mathfrak{g},K)} \arrow[r, "h^{cl}"] & \Gamma(\mathcal{Y}, P_{\mathcal{Y}}) \end{tikzcd}$$

Need $\mathcal{Y}$ is good, which is satisfied if K of finite type, or $\mathcal{Y} = \mathrm{Bun}_G(X)$, genus$(X)>1$.

local   global

Def • Local quantization condition is that $\sigma_{(\mathfrak{g},K)}$ is an isomorphism.
• Global quantization condition is that h is strictly compatible with filtrations, namely if $h: A \to B$, with $F^\bullet, G^\bullet$ filtrations on A, B, then $h(F^i A) = G^i B \cap h(A)$.

If both local & global quantization met, then

$$\sigma_{\mathcal{Y}} : \mathrm{gr}(h(D_{(\mathfrak{g},K)})) \xrightarrow{\sim} h^{cl}(P_{(\mathfrak{g},K)}).$$

Remarks: • All constructions may be twisted to a central extension $(\tilde{\mathfrak{g}}, K)$ of $(\mathfrak{g}, K)$, and replacing $D_{\mathcal{Y}}$ by $D_{\mathcal{Y},\mathcal{L}}$ for some line bundle $\mathcal{L}$ on $\mathcal{Y}$

• We assumed K of fin. type for simplicity, but there is natural upgrade to affine gp schemes, e.g. $K = G(\mathcal{O})$.

II | The Hitchin map construction.

Let G = semisimple group $/\mathbb{C}$

X = smooth projective curve of genus $g > 1$ ($\Rightarrow \mathrm{Bun}_G$ good)

$$C := \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{g})^G \overset{\text{Non-canonically}}{\cong} \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{h})^W,$$ hence is affine

$$C_{\Omega_X} := \Omega_X \times_{\mathbb{C}^*} C,$$ $\mathbb{C}^*$ acts by scalar mult. on $\mathfrak{g}^*$ & $\mathbb{C}^* \curvearrowright \Omega_X$ since it's a line bundle.

view as bundle over X

$$\mathrm{Hitch}(X) := \Gamma(X, C_{\Omega_X})$$ = affine space of dim $(g-1)\cdot \dim G$

$$\mathfrak{z}^{cl} := \mathcal{O}(\mathrm{Hitch}(X))$$ = graded (by $\mathbb{G}_m$-action) commutative algebra.

"exponents" of $\mathfrak{g}$.

We have (non-canonical) isom $\mathrm{Hitch}(X) \simeq \prod_{i \in I} \Gamma(X, \Omega_X^{\otimes d_i})$

Fix $P \in \mathrm{Bun}_G(X)$. The map $\mathfrak{g}^* \longrightarrow C$

induces $\mathfrak{g}^*_P = \mathfrak{g}^* \times_G P \longrightarrow C \times X \longrightarrow C \times_{\mathbb{C}^*} X$,

a map of bundles over X. Twist by Ω_X & take global sections:

$$\mu_P : \Gamma(X, \mathfrak{g}^*_P \otimes \Omega_X) \longrightarrow \Gamma(X, C \times_{\mathbb{C}^*} \Omega_X) = \mathrm{Hitch}(X)$$

By Serre duality & Ginzburg's lecture,

$$H^0(X, \mathfrak{g}^*_P \otimes \Omega_X) \simeq H^1(X, \mathfrak{g}_P)^* \simeq T^*_P \mathrm{Bun}_G.$$

So, we have a map

$$\mu_P : T^*_P \mathrm{Bun}_G \longrightarrow \mathrm{Hitch}(X).$$

In fact, this glues over $P \in \mathrm{Bun}_G$, and is a <u>map of stacks</u>

(compatibility w/ smooth test schemes S, not just $\mathbb{C}$-points)

$$\boxed{\Longrightarrow \mu : T^*\mathrm{Bun}_G \longrightarrow \mathrm{Hitch}(X),}$$

called the <u>global Hitchin map</u>

- See notes for Ginzburg's talk Ex. 3.2 for explicit construction of μ in the $G = GL_n(\mathbb{C})$ - case.

We also get a morphism of <u>graded commutative algebras</u>

$$\implies h^{cl} : \mathfrak{z}^{cl} \longrightarrow \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}_{T^*\mathrm{Bun}_G})$$

$$\boxed{\begin{array}{l}\underline{\text{Main Thm}} \\ \text{The image of } h^{cl} \text{ consists of Poisson commuting functions.}\end{array}}$$

- <u>Remark</u>: Another construction of Hitchin map (avoiding a "patching argument"):

$$\tilde{\pi} : \mathfrak{g}/G \longrightarrow \mathfrak{g}/\!/G.$$

Twist by Ω_X, viewed as $\mathbb{C}^*$ - torsor over X:

$$\tilde{\pi} : \Omega_X \times_{\mathbb{C}^*} \mathfrak{g}/G \longrightarrow \Omega_X \times_{\mathbb{C}^*} \mathfrak{g}/\!/G,$$

Finally, take the stack of sections over X:

$$\pi : \underbrace{\mathrm{Maps}_X(X, \Omega_X \times_{\mathbb{C}^*} \mathfrak{g}/G)}_{= \, T^*\mathrm{Bun}_G} \longrightarrow \underbrace{\mathrm{Maps}_X(X, \Omega_X \times_{\mathbb{C}^*} \mathfrak{g}/\!/G)}_{= \, \mathrm{Hitch}(X).}$$

Check on fibers using Serre duality.

<u>Recall</u> Ginzburg / Hitchin's conjecture: The Schouten bracket on $\mathrm{Im}(\Lambda^\bullet h^{cl}) \subseteq \Gamma(\Lambda^\bullet \mathcal{O}_{T^*\mathrm{Bun}_G})$ also commutes

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BREAK

Hitchin gave complex-analytic proof, [BD] gave alg. pf, which we follow.

<u>pf</u> We first need a "local" construction of the Hitchin map.

Fix closed point $x \in X$ & consider

$$\hat{\mathcal{O}}_x := \varprojlim_n \mathcal{O}_x / m_x^n \quad (\simeq \mathbb{C}[[t]]) \quad \text{Formal disk.}$$

$$C_{\Omega_{\hat{\mathcal{O}}_x}} := C \times_{\mathbb{C}^*} \Omega_{\hat{\mathcal{O}}_x}, \quad \Omega_{\hat{\mathcal{O}}_x} \simeq k[[t_x]]\, dt_x .$$

$$\mathrm{Hitch}(X) = \Gamma(X, C_{\Omega_X}) \xhookrightarrow[\text{embedding b/c sections of a bundle determined by its Taylor expansion at } x.]{} \Gamma(D_x, C_{\Omega_{\hat{O}_x}}) =: \mathrm{Hitch}_x(X) \quad (*)$$

Let $\mathfrak{z}^{cl}_x = \mathcal{O}(\mathrm{Hitch}_x(X)) \Rightarrow$ Get <u>surjective</u> morphism $\theta^{cl}: \mathfrak{z}^{cl}_x \twoheadrightarrow \mathfrak{z}^{cl}$

$$\begin{tikzcd}
\mathfrak{z}^{cl}_x \arrow[r, two heads, "\theta^{cl}"] \arrow[dr, "h^{cl}_x"'] & \mathfrak{z}^{cl} \arrow[d, "h^{cl}"] \\
& \Gamma(T^*\mathrm{Bun}_G, \mathcal{O})
\end{tikzcd}$$

Remark: $\mathrm{Hitch}_x(X) \overset{\text{non canonically}}{\simeq} \prod_{i \in I} \Omega_{\hat{O}_x}^{\otimes d_i}$, where $\Omega_{\hat{O}_x} \simeq k[[t]]\,dt$, so $(*)$ is just $\Omega_X \xrightarrow{\text{restrict}} \Omega_{\hat{O}_x}$.

Thus suffices to show $\mathrm{Im}(h^{cl}_x)$ Poisson commutes.

We show this using <u>the Harish-Chandra pair</u>:

$$Y = \mathrm{Bun}_G^{(\infty x)} := \left\{ (P, \tau_x) \;\middle|\; P \in \mathrm{Bun}_G,\ \tau: P|_{D_x} \simeq P^{triv}|_{D_x} \right\}$$

<u>Fact</u>: $Y \simeq \varprojlim_n \mathrm{Bun}_G^{(nx)}$ is a scheme & a $G(\hat{O}_x)$-torsor over Bun_G.

$K = G(\hat{O}_x)$

$y = \mathrm{Bun}_G$

$\mathfrak{g} = \mathfrak{g} \otimes \hat{K}_x \cong \mathfrak{g}((t))$

Geometrically, we have a commutative diagram (of stacks) over X

$$\begin{tikzcd}
T^*\mathrm{Bun}_G^{(\infty x)} \arrow[r, "h^{cl}_1"] & \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}_{\hat{K}_x})) \\
T^*\mathrm{Bun}_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{(\infty x)} \arrow[u, hook, "\text{closed embedding}"] \arrow[r, "h^{cl}_2"] \arrow[d, "\text{dominant (dense image)}"'] & \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}_{\hat{K}_x}/\mathfrak{g}_{\hat{O}_x})) \arrow[u, hook, "\text{closed embedding}"'] \arrow[d, "\text{(dense image) dominant}"] \\
T^*\mathrm{Bun}_G \arrow[r, "h^{cl}_3"] & \mathrm{Spec}\left(\mathrm{Sym}(\mathfrak{g}_{\hat{K}_x}/\mathfrak{g}_{\hat{O}_x})^{G(\hat{O}_x)}\right)
\end{tikzcd}$$

$\mathrm{Sym}(\mathfrak{g}_{\hat{K}_x}/\mathfrak{g}_{\hat{O}_x}) = P_{(\mathfrak{g},K)}$ in §1 notation.

where h_i^{cl} are induced by the <u>Hamiltonian actions</u> of

$h_1^{cl} : \mathfrak{g} \curvearrowright Y$, h_2^{cl} is restriction of h_1^{cl},

$h_3^{cl} : (\mathfrak{g}, K) \curvearrowright \mathcal{Y} = K \backslash Y$

$\varprojlim_n T_p \mathrm{Bun}_G^{(nx)} \simeq \varprojlim_n H^0(X, \mathfrak{g}_P^*(-nx) \otimes \Omega_X)$

Explicitly, $T^* \mathrm{Bun}_G^{(\infty x)} = \left\{ (P, \tau_x, f) \;\middle|\; (P, \tau_x) \in \mathrm{Bun}_G^{(\infty x)},\ \overbrace{f \in \Gamma(X \setminus x, \mathfrak{g}_P^* \otimes \Omega_X)}^{\parallel} \right\}$

$$\begin{array}{ccc} T^*\mathrm{Bun}_G^{(\infty x)} & & (P, \tau_x, f) \\ \Big\downarrow h_1^{cl} & & \Big\downarrow \\ \mathrm{Spec}(\mathrm{Sym}\, \mathfrak{g}_{\hat{K}_x}) & & \left(\underset{\mathfrak{g}_{\hat{K}_x}}{z \in} \longmapsto \mathrm{Res}_x (f|_{D_x^\times}, z) \right) \in \mathfrak{g}^*_{\hat{K}_x} \end{array}$$

Also, have commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}_{\hat{K}_x})) \arrow[r, "\chi_1", "\sim"'] & \Gamma(D_x^\times, \mathfrak{g}^*_{\Omega_{K_x}}) \\ \mathrm{Spec}\, \mathrm{Sym}(\mathfrak{g}_{\hat{K}_x} / \mathfrak{g}_{\hat{O}_x}) \arrow[r, "\chi_2", "\sim"'] \arrow[u, hook] \arrow[d, twoheadrightarrow] & \Gamma(D_x, \mathfrak{g}^*_{\Omega_{O_x}}) \arrow[u, hook] \arrow[d, twoheadrightarrow] \\ \mathrm{Spec}\, \mathrm{Sym}(\mathfrak{g}_{\hat{K}_x} / \mathfrak{g}_{\hat{O}_x})^{G(\hat{O}_x)} \arrow[r, "\chi_3"] & \Gamma(D_x, C_\Omega) = \mathrm{Hitch}_x X \end{tikzcd}$$

where $\chi_1 : f \longmapsto (\omega \longmapsto \mathrm{Res}_x(f, \omega))$ is an isomorphism

χ_2 is restriction of χ_1, can show is an isomorphism.

χ_3 is the canonically induced map on the (GIT) quotient

<u>Lemma</u> χ_3 is isomorphism.

<u>Pf</u> First, we show χ_3 is closed embedding:

The residue pairing identifies $(K_x / O_x)^*$ with Ω_{O_x}:

So, $\begin{cases} \mathrm{Spec}\left(\mathrm{Sym}(\mathfrak{g}_x/\mathfrak{g}_0)\right) = \mathfrak{g}^* \otimes \Omega_{\mathcal{O}_x} \simeq \mathfrak{g}^* \otimes \mathcal{O}_x\, dt_x \\ \mathrm{Hitch}_x X = \prod_i \Omega_{\mathcal{O}_x}^{\otimes d_i} \simeq \prod_i \mathcal{O}_x^{\otimes d_i}\, dt_x . \end{cases}$

Thus, may replace:

$$\begin{cases} \mathfrak{g}^* \otimes \Omega_{\mathcal{O}_x} & \text{with} \quad \mathfrak{g}^* \otimes \mathcal{O}_x = \mathrm{Mor}_{Sch}(D_x, \mathfrak{g}^*) \\ \mathrm{Hitch}_x X & \text{with} \quad \mathrm{Mor}_{Sch}(D_x, C) \end{cases}$$

$= \mathfrak{g}^*/\!/G$

and we just want to show:

$$\mathrm{Fun}(\mathrm{Mor}(D_x, C), \mathbb{C}) \longrightarrow \mathrm{Fun}(\mathrm{Mor}(D_x, \mathfrak{g}^*), \mathbb{C})^{G(\mathcal{O}_x)}$$

is surjective

Since replacing a space by open subset with complement of codim ≥ 2 does not affect H^0, we may replace $\mathfrak{g}$ by $\mathfrak{g}^{reg}$.

But now the projection map p

$$\mathfrak{g}^*_{reg} \longrightarrow C$$

is smooth, surjective, & the G-action on fibers is transitive!

It follows $\varkappa_3$ is closed embedding. In fact, since $\mathfrak{g}^*_{reg} \to \mathfrak{g}^*_{reg}/G$ is smooth we may lift any arc $\in \mathrm{Mor}(D_x, \mathfrak{g}^*_{reg}/G)$ along this projection (formal smoothness argument) to deduce $\varkappa_3$ in fact contains the dense subscheme $\mathrm{Mor}(D_x, \mathfrak{g}^*_{reg}/G) \subseteq \mathrm{Mor}(D_x, \mathfrak{g}^*/G)$. ∎ can also see dominance using Kostant slice!

Corollary $\mathrm{Im}(h_x^{cl})$ Poisson commutes.

Pf First observe the local Hitchin map factors as Poisson algebra maps:

$$\begin{tikzcd} \mathcal{O}(\mathrm{Hitch}_x X) =: \mathfrak{Z}_x^{cl} \arrow[rr, "h_x^{cl}"] \arrow[dr, "\varkappa_3", "\sim"'] & & \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}_{T^*\mathrm{Bun}_G}) \\ & P_{(\mathfrak{g}, K)} \arrow[ur, "h_3"'] & \end{tikzcd} \qquad (\dagger)$$

h_3 preserves Poisson bracket $\Rightarrow$ suffices $\mathrm{Im}(\mathcal{X}_3)$ Poisson commutes. We can complete, to make this statement trivial:

$$\bar{a}^{cl} : \overline{\mathrm{Sym}(\mathfrak{g}_{K_x})}^{\mathfrak{g}_{K_x}} \longrightarrow \mathrm{Sym}(\mathfrak{g}_{K_x}/\mathfrak{g}_{O_x})^{G(O_x)} = P_{(\mathfrak{g},K)}$$

is clearly surjective & the source is the Poisson center of $\overline{\mathrm{Sym}}(\mathfrak{g}_K)$,

(can directly write a preimage, see BD 2.4.2)

hence the Poisson bracket on target is 0 & we're done!

<u>Note</u>: to see commutativity of $(\dagger)$, we use

(1) $\mathcal{X}_1 \circ h_1^{cl} : (P, \tau_x, f) \longmapsto (\omega \mapsto \mathrm{Res}_x(f|_{D_x^\times}, \omega)) \longmapsto f|_{D_x^\times}$

$\Rightarrow$ (2) $\mathcal{X}_2 \circ h_2^{cl} : (P, \tau_x, f) \longmapsto f|_{D_x}$

since $f \in \Gamma(X \setminus x, \mathfrak{g}_P^* \otimes \Omega_X)$ no longer has poles

$\Rightarrow$ (3) $\mathcal{X}_3 \circ h_3^{cl} : (P, f) \longmapsto f|_{D_x}$ mod $G(O_x)$ $\blacksquare$

III) A glimpse to the future: (aka: singular support of Hecke eigensheaf)

Assuming we may find a global quantization of the classical Hitchin fibration $\mu : T^*\mathrm{Bun}_G \longrightarrow \mathrm{Hitch}(X)$, we may show the Hecke eigensheaf is non zero:

<u>Thm</u> (Global quantization exists, proved in talk 8)

There exists a filtered commutative algebra $\mathfrak{z}(X)$ such that $\mathrm{gr}(\mathfrak{z}(X)) \simeq \mathfrak{z}^{cl}(X)$, and there exists a map

$$h : \mathfrak{z}(X) \longrightarrow \Gamma(\mathrm{Bun}_G, D')$$

twisted diff. ops. on Bun_G

such that

$$\begin{tikzcd}
\mathrm{gr}\, \mathfrak{z}(X) \arrow[r, "\mathrm{gr}(h)"] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{gr}\, \Gamma(\mathrm{Bun}_G, D') \arrow[d, "\wr"] \\
\mathfrak{z}^{cl}(X) \arrow[r, "h^{cl}"] & \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}_{T^*\mathrm{Bun}_G})
\end{tikzcd}$$

Let $Op_{{}^LG}(X) = \mathrm{Spec}\, \mathfrak{z}(X)$, $\sigma \in Op_{{}^LG}(X)$, and consider

$$D'_\sigma := D' \otimes_{\mathfrak{z}(X)} k_\sigma \in D'\text{-mod}(\mathrm{Bun}_G),$$

called the Hecke eigensheaf

where $k_\sigma = \mathfrak{z}(X)/m_\sigma$.

Thm The singular support, $SS(D'_\sigma)$, equals $\overbrace{\mu^{-1}(0)}^{\text{"Global nilpotent cone"}}$,
where $\mu : T^*\mathrm{Bun}_G \to \mathrm{Hitch}(X)$ is Hitchin fibration. As a corollary,

D'_σ is holonomic, and non zero.

Pf It is known μ is flat & its fibers are all of dim = dim Bun_G.
(Ref: Ginzburg - Global nilpotent cone is Lagrangian), hence

$\mathrm{gr}(D')$ is flat over $\mathrm{gr}(\mathfrak{z}(X)) = \mathfrak{z}^{cl}(X)$.

$\Rightarrow D'$ is flat over $\mathfrak{z}(X)$. Let $m_\sigma \subseteq \mathfrak{z}(X)$ be max ideal corresponding to σ.

$$\Rightarrow \mathrm{gr}(D'_\sigma) = \mathrm{gr}(D'/D'\cdot m_\sigma) = \mathrm{gr}(D')/\mathrm{gr}(D')\cdot \mathrm{gr}(m_\sigma)$$
$$= \mathcal{O}(T^*\mathrm{Bun}_G)/\mathcal{O}(T^*\mathrm{Bun}_G)\cdot \mathrm{gr}(m_\sigma)$$

$$\Rightarrow SS(D'_\sigma) = \mu^{-1}(0)$$ ∎